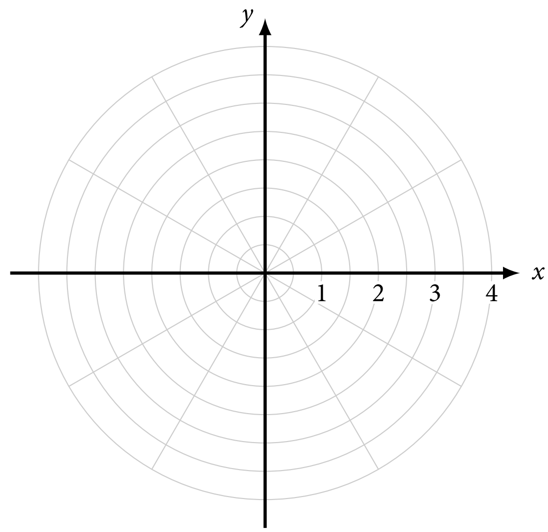
Lesson 32. Double Integrals in Polar Coordinates

1 Review

1.1 Polar coordinates

- **Polar coordinate system**: specify points in the xy-plane as (r, θ) where
  - $r =$
  - $\theta =$

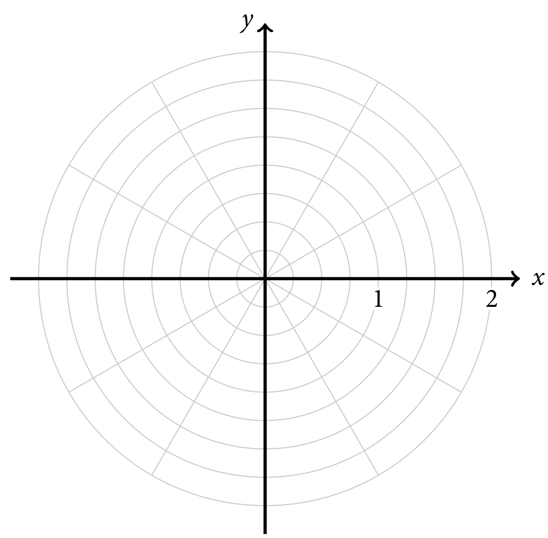
Example 1. Sketch the region in the plane consisting of points whose polar coordinates satsify: $1 \leq r \leq 3$, $\pi/6 \leq \theta \leq 5\pi/6$.



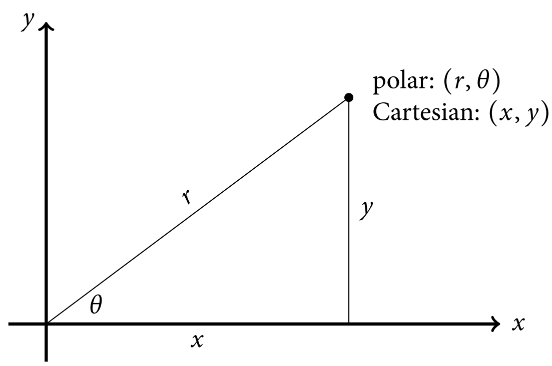
1.2 Polar curves

- The **graph of a polar equation** $F(r, \theta) = 0$ consists of all points that can be represented by some polar coordinates (r, θ) that satisfy the equation

Example 2. Sketch the curve with polar equation $r = 2\cos\theta$.



1.3 Correspondence between polar and Cartesian coordinates



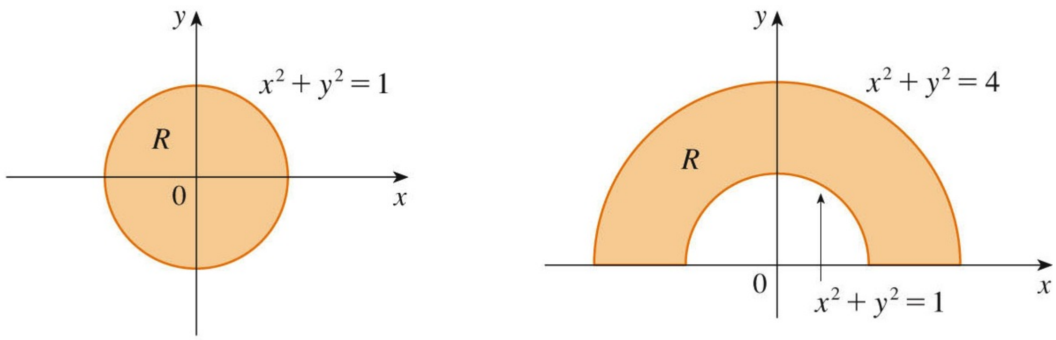
- $x =$
- $y =$
-
-

Example 3. Find a Cartesian equation for the curve $r = 2\cos\theta$.

Example 4. Find a polar equation for the curve represented by the Cartesian equation $4y^2 = x^2$.

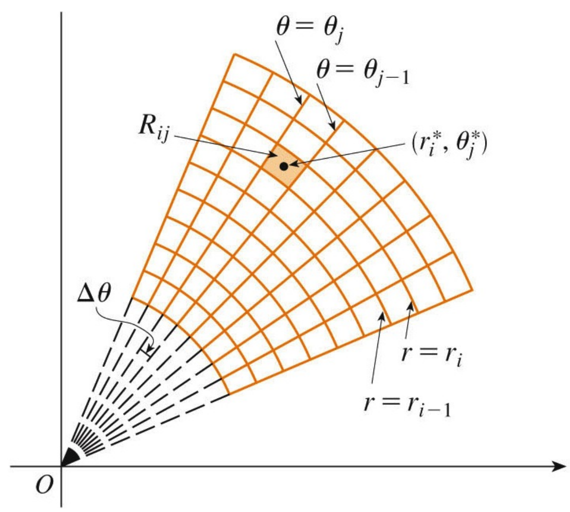
2 Changing to polar coordinates in a double integral

- Idea:
  - Some regions are hard to express in terms of rectangular coordinates, but easily described using polar coordinates



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

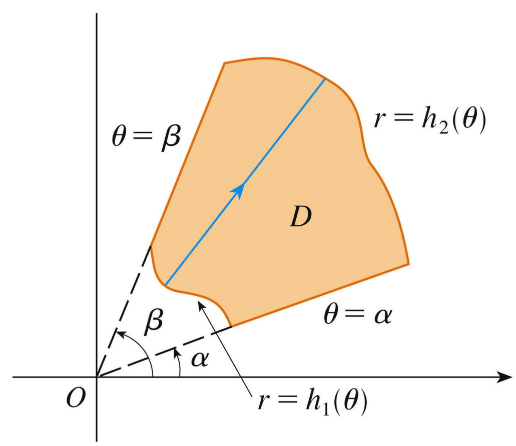
  - How do we integrate in polar coordinates? Divide regions into **polar subrectangles**



- If D is a polar region of the form

$$D = \left\{(r, \theta) \,|\, \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\right\}$$

then



- Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into $f(x, y)$
- Replace dA with $r\,dr\,d\theta$
- **Don't forget the additional factor r!**

Example 5. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.

Example 6. Evaluate $\iint_D (x^2+y^2)\,dA$, where D is the region in the first quadrant bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$ and the lines $x=0$ and $y=x$.

Example 7. Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.